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Space-Filling Curves and Their Applications in Scientific Computing

Space-Filling Curves

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Classification of Space-filling Curves

Definition: (recursive space-filling curve)

A space-filling curve $f : I \rightarrow Q \subset \mathbb{R}^n$ is called recursive, if both $I$ and $Q$ can be divided in $m$ subintervals and subdomains, such that

- $f_\ast(I(\mu)) = Q(\mu)$ for all $\mu = 1, \ldots, m$, and
- all $Q(\mu)$ are geometrically similar to $Q$.

Definition: (contiguous space-filling curve)

A recursive space-filling curve is called contiguous, if for any two neighbouring intervals $I(\nu)$ and $I(\mu)$ also the corresponding subdomains $Q(\nu)$ and $Q(\mu)$ are direct neighbours, i.e. share an $(n - 1)$-dimensional hyperplane.
Contiguous, Recursive Space-filling Curves

Examples:
- all Hilbert curves (2D, 3D, …)
- also Peano curves and Sierpinski curves

Properties: contiguous, recursive SFC are
- continuous (more exact: Hölder continuous with exponent $1/n$)
- neighbourship-preserving
- describable by a grammar
- describable in an arithmetic form
  (see full set if slides)
3D Hilbert Curves

• Wanted: contiguous, recursive SFC, based on division-by-2
  ⇒ leads to 3 basic patterns:

• in addition: symmetric forms, change of orientation
• always two different orientations of the components
  ⇒ numerous different Hilbert curves
3D Hilbert Curves – Iterations

1st iteration

2nd iteration
3D Hilbert Curves – Variants

Different orientation of the sub-cubes:

- same basic pattern (“Motiv”), same approximating polygon
- differences only visible from 2nd iteration
Parallelisation using Space-filling Curves

Problem setting:
- “mesh” (2D, 3D, ...) of $N$ unknowns ($N \gg 1000$)
- solve linear system(s) of equations (maybe repeatedly with varying right-hand side)
- in the system, only spatially neighbouring unknowns are coupled

Parallelisation:
Distribute $N$ unknowns to $p$ partitions, such that
- each partition contains the same number of unknowns (load balancing)
- for as many unknowns as possible, all neighbours are in the same partition (⇒ avoids communication between partitions)
Parallelisation using Space-filling Curves (2)

Further demand: adaptivity

- add further unknowns (during/depending on intermediate results) or drop unknowns
- (re-)partitioning required to be fast: must not cost more computation time than going on with a bad load balance
- “shape preserving”: if only few unknowns are added or dropped, the shape of partitions should not change strongly
  ⇒ only few unknowns then need to migrate to another partition

⇒ popular strategy: use space-filling curves
Hölder Continuity of Space-filling Curves

Definition: (Hölder continuous)

A function $f$ is called Hölder continuous with exponent $r$ on the interval $I$, if a constant $C > 0$ exists, such that for all $x, y \in I$:

$$\| f(x) - f(y) \|_2 \leq C |x - y|^r$$

Importance for space-filling curves:

- $|x - y|$ is the distance of the indices
- $\| f(x) - f(y) \|$ is the distance of the image points (in “space”)
- To prove: the Hilbert curve is Hölder continuous with exponent $r = d^{-1}$, where $d$ is the dimension:

$$\| f(x) - f(y) \|_2 \leq C |x - y|^{1/d} = C \sqrt[2]{x - y}$$
Hölder Continuity of the 3D Hilbert Curve

Proof analogous to simple continuity proof:

• given $x, y \in I$; find an $n$, such that $8^{-(n+1)} < |x - y| < 8^{-n}$
• $8^{-n}$ is the interval length for the $n$-th iteration
  $\Rightarrow [x, y]$ covers at most two neighbouring(!) intervals.
• per construction of the 3D Hilbert curve, the function values $h(x)$
  and $h(y)$ are in two adjacent cubes of side length $2^{-n}$.
• the length of the space diagonal through the two adjacent cubes
  is $2^{-n} \cdot \sqrt{1^2 + 1^2 + 2^2} = 2^{-n} \cdot \sqrt{6}$, hence:

$$\|h(x) - h(y)\|_2 \leq 2^{-n} \sqrt{6} = (8^{-n})^{1/3} \sqrt{6} = \left(8^{-(n+1)}\right)^{1/3} \sqrt{6}^{1/3} \sqrt{6}$$
$$\leq 2 \sqrt{6} |x - y|^{1/3} \quad \text{q.e.d.}$$
Hölder Continuity and Parallelisation

- for the Hilbert curve (also Peano curve and all contiguous, recursive SFC), we have:

\[ \|f(x) - f(y)\|_2 \leq C^d \sqrt{|x - y|} \]

- relates the distance \(|x - y|\) between indices to the distance \(\|f(x) - f(y)\|\) of (mesh) points

- \(|x - y|\) corresponds to the section covered by the SFC (\(\hat{=}\) area/volume)

- gives relation between volume (number of grid cells/points) and extent (e.g. radius) of a partition

\[ \Rightarrow \] Hölder continuity gives a quantitative estimate for compactness of partitions
Locality and Cartesian Grids

• from the Hölder continuity

\[ \|f(x) - f(y)\| \leq C \sqrt[\alpha]{|x - y|} \]

we directly obtain the following estimate:

\[ \frac{\|f(x) - f(y)\|^d}{|x - y|} \leq \hat{C} \]

• for a discrete (cartesian, e.g.) grid, we can compare this to the relation

\[ \frac{\|x_i - x_j\|^d}{|i - j|} \leq \hat{C}, \]

where \( i \) and \( j \) are the indices (memory positions, e.g.) of two grid cells at positions \( x_i \) and \( x_j \).