ACOPhys – State University of St. Petersburg

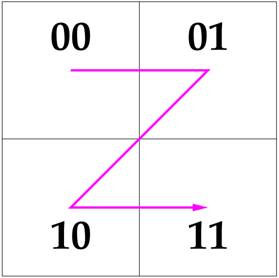
# Space-Filling Curves and Their Applications in Scientific Computing

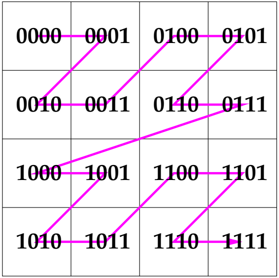
## Space-Filling Curves

Michael Bader

Technische Universität München, Sep 1–5, 2008

# Start: Morton Order / Cantor's Mapping




## Questions:

- Can this mapping lead to a **contiguous** "curve"?
- i.e.: Can we find a **continuous** mapping?
- and: Can this continuous mapping fill the entire square?

# What is a Curve?

## Definition (Curve)

As a *curve*, we define the image $f_*(\mathcal{I})$ of a continuous mapping $f\colon \mathcal{I} \to \mathbb{R}^n$.
$x = f(t), t \in \mathcal{I}$ is called *parameter representation* of the curve.

With:

- $\mathcal{I} \subset \mathbb{R}$ and $\mathcal{I}$ is compact, usually $\mathcal{I} = [0, 1]$.
- the *image* $f_*(\mathcal{I})$ of the mapping $f_*$ is defined as $f_*(\mathcal{I}) := \{f(x) \in \mathbb{R}^n \mid x \in \mathcal{I}\}$.
- $\mathbb{R}^n$ may be replaced by any Euklidian vector space (norm & scalar product required).

# What is a Space-filling Curve?

## Definition (Space-filling Curve)

Given a mapping $f\colon \mathcal{I} \to \mathbb{R}^n$, then the corresponding curve $f_*(\mathcal{I})$ is called a *space-filling curve*, if the Jordan content (area, volumne, ...) of $f_*(\mathcal{I})$ is larger than 0.

Comments:

- assume $f\colon \mathcal{I} \to \mathcal{Q} \subset \mathbb{R}^n$ to be *surjective* (i.e., every element in $\mathcal{Q}$ occurs as a value of $f$;

  then, $f_*(\mathcal{I})$ is a space-filling curve, if the area (volume) of $\mathcal{Q}$ is positive.
- if the domain $\mathcal{Q}$ has a smooth boundary, then there can be *no bijective mapping* $f\colon \mathcal{I} \to \mathcal{Q} \subset \mathbb{R}^n$, such that $f_*(\mathcal{I})$ is a space-filling curve
  (theorem: E. Netto, 1879).

# History of Space-Filling Curves

**1877:** Georg Cantor finds a bijective mapping from the unit interval $[0,1]$ into the unit square $[0,1]^2$.

**1879:** Eugen Netto proves that a *bijective* mapping $f\colon \mathcal{I} \rightarrow \mathcal{Q} \subset \mathbb{R}^n$ can not be continuous (i.e., a curve) at the same time (as long as $\mathcal{Q}$ has a smooth boundary).

**1886:** rigorous definition of *curves* introduced by Camille Jordan

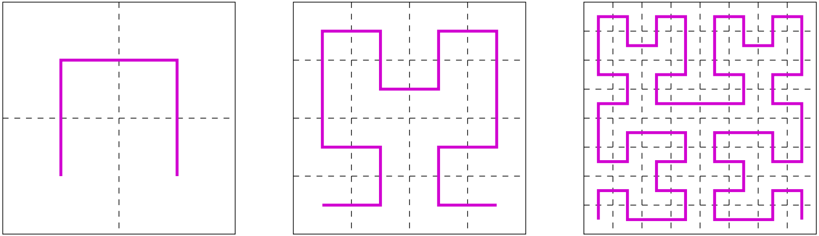
**1890:** Giuseppe Peano constructs the first space-filling curves.

**1890:** Hilbert gives a geometric construction of Peano's curve; and introduces a new example – the Hilbert curve

**1904:** Lebesgue curve

**1912:** Sierpinski curve

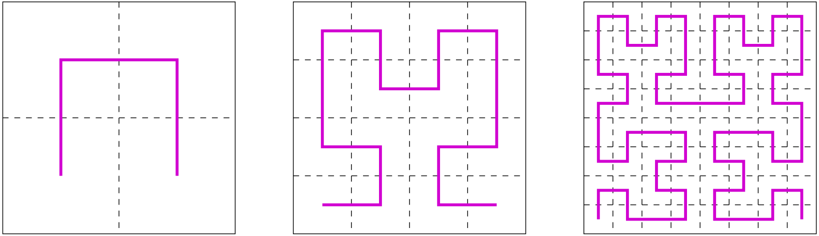
# Construction of the Hilbert curve



*Iterations* of the Hilbert curve:

- start with an iterative numbering of 4 subsquares
- combine four numbering patterns to obtain a twice-as-large pattern
- proceed with further iterations

# Construction of the Hilbert curve

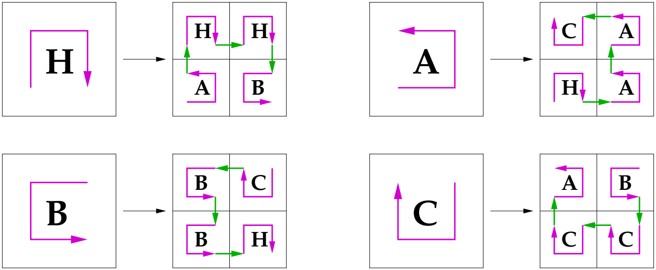


Recursive construction of the *iterations*:

- split the quadratic domain into 4 congruent subsquares
- find a space-filling curve for each subdomain
- join the four subcurves in a suitable way

# A Grammar for Describing the Hilbert Curve

Construction of the iterations of the Hilbert curve:




→ motivates a **Grammar** to generate the iterations

# A Grammar for Describing the Hilbert Curve

- Non-terminal symbols: $\{H, A, B, C\}$, start symbol $H$
- terminal characters: $\{\uparrow, \downarrow, \leftarrow, \rightarrow\}$
- productions:

$$
\begin{aligned}
H &\longleftarrow A \uparrow H \rightarrow H \downarrow B \\
A &\longleftarrow H \rightarrow A \uparrow A \leftarrow C \\
B &\longleftarrow C \leftarrow B \downarrow B \rightarrow H \\
C &\longleftarrow B \downarrow C \leftarrow C \uparrow A
\end{aligned}
$$

- replacement rule: in any word, **all non-terminals have to be replaced at the same time** $\rightarrow$ L-System (Lindenmayer)

$\Rightarrow$ the arrows describe the **iterations of the Hilbert curve** in “turtle graphics”

# Definition of the Hilbert Curve's Mapping

**Definition:** (Hilbert curve)

- each parameter $t \in \mathcal{I} := [0, 1]$ is contained in a sequence of intervals
$$\mathcal{I} \supset [a_1, b_1] \supset \ldots \supset [a_n, b_n] \supset \ldots ,$$
where each interval result from a division-by-four of the previous interval.
- each such sequence of intervals can be uniquely mapped to a corresponding sequence of 2D intervals (subsquares)
- the 2D sequence of intervals converges to a unique point $q$ in $q \in \mathcal{Q} := [0, 1] \times [0, 1]$ – $q$ is defined as $h(t)$.

**Theorem**

*$h : \mathcal{I} \to \mathcal{Q}$ defines a space-filling curve, the Hilbert curve.*

# Proof: $h$ defines a Space-filling Curve

We need to prove:

- $h$ is a mapping, i.e. each $t \in \mathcal{I}$ has a *unique* function value $h(t)$
  $\rightarrow$ OK, if $h(t)$ is independent of the choice of the sequence of intervals
- $h \colon \mathcal{I} \rightarrow \mathcal{Q}$ is *surjective*:
  - for each point $q \in \mathcal{Q}$, we can construct an appropriate sequence of 2D-intervals
  - the 2D sequence corresponds in a unique way to a sequence of intervals in $\mathcal{I}$ – this sequence defines an original value of $q$
    $\Rightarrow$ every $q \in \mathcal{Q}$ occurs as an image point.
- $h$ is *continuous*

# Continuity of the Hilbert Curve

A function $f: \mathcal{I} \rightarrow \mathbb{R}^n$ is *continuous*, if

for each $\epsilon > 0$

a $\delta > 0$ exists, such that

for all $t_1, t_2 \in \mathcal{I}$ with $|t_1 - t_2| < \delta$, the following inequality holds:

$\|f(t_1) - f(t_2)\|_2 < \epsilon$

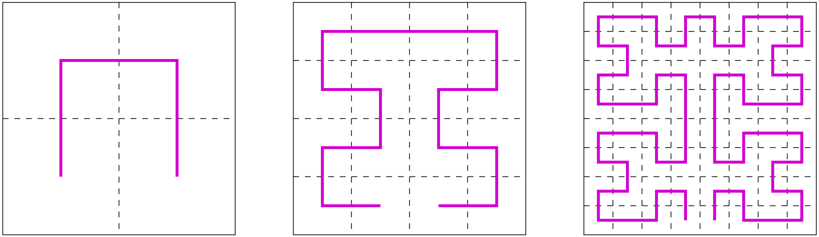
**Strategy for the proof:**
For any given parameters $t_1, t_2$, we compute an estimate for the disctance $\|h(t_1) - h(t_2)\|_2$ (functional dependence on $|t_1 - t_2|$).
$\Rightarrow$ for any given $\epsilon$, we can then compute a suitable $\delta$

# Continuity of the Hilbert Curve (2)

- given: $t_1, t_2 \in \mathcal{I}$; choose an $n$, such that $|t_1 - t_2| < 4^{-n}$
- in the $n$-th iteration of the interval sequence, all interval are of length $4^{-n}$
  $\Rightarrow [t_1, t_2]$ overlaps at most two neighbouring(!) intervals.
- due to construction of the Hilbert curve, the values $h(t_1)$ and $h(t_2)$ will be in neighbouring subsquares with face length $2^{-n}$.
- the two neighbouring subsquares build a rectangle with a diagonal of length $2^{-n} \cdot \sqrt{5}$;
  therefore: $\|h(t_1) - h(t_2)\|_2 \leq 2^{-n}\sqrt{5}$

For a given $\epsilon > 0$, we choose an $n$, such that $2^{-n}\sqrt{5} < \epsilon$.
Using that $n$, we choose $\delta := 4^{-n}$; then, for all $t_1, t_2$ with $|t_1 - t_2| < \delta$, we get: $\|h(t_1) - h(t_2)\|_2 \leq 2^{-n}\sqrt{5} < \epsilon$. Which proves the continuity!

# Construction of the Hilbert-Moore Curve



New Construction:

- modified orientation of the subcurves in the first iteration
- leads to a closed curve: start and end point at $\left(0, \frac{1}{2}\right)$