ACOPhys – State University of St. Petersburg

Space-Filling Curves and Their Applications in Scientific Computing

From Quadtrees to Space-Filling Curves

Michael Bader

Technische Universität München, Sep 1–5, 2008

Overview: Modelling of Geometric Objects

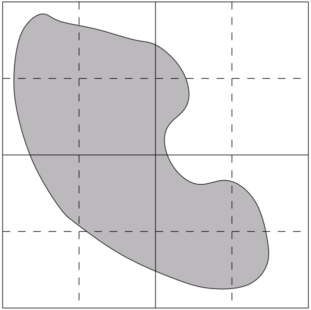
Surface-oriented models:

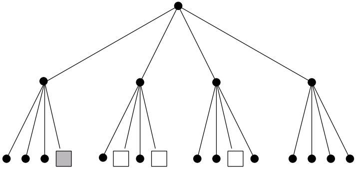
- wire-frame models
- augmented models using Bezier curves and planes
- typically described by graphs on nodes, edges, and faces

Volume-oriented models:

- Constructive Solid Geometry (boolean operations on primitives)
- voxel models: place object in a grid
- octrees: recursive refinement of voxel grids

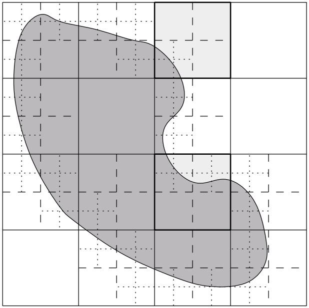
Quadtrees to Describe Geometric Objects

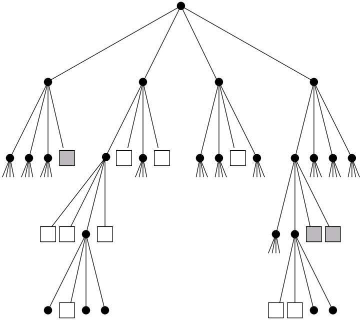




- start with an initial square (covering the entire domain)
- recursive substructuring into four subsquares

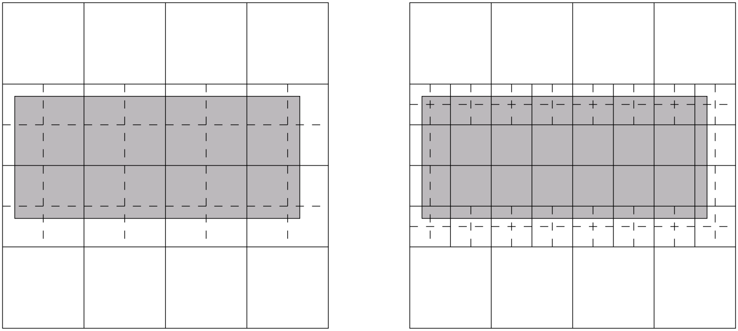
Quadtrees to Describe Geometric Objects





- start with an initial square (covering the entire domain)
- recursive substructuring into four subsquares
- adaptive refinement possible
- terminate, if squares entirely within or outside domain

Number of Quadtree Cells to Store a Rectangle

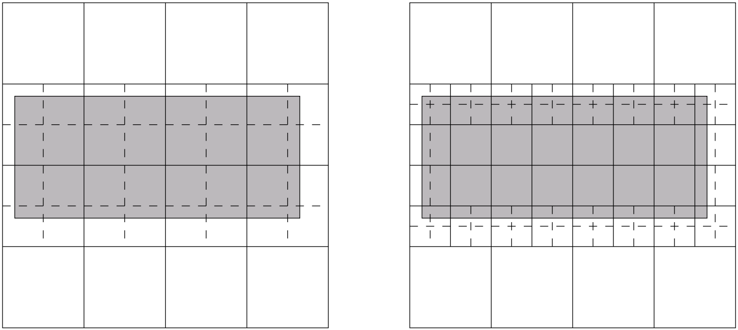


Terminal (s_k) and boundary (r_k) cells after k refinement steps:

$$r_k \approx 2 \cdot r_{k-1} \qquad \qquad r_k \approx 2^{k+1}$$

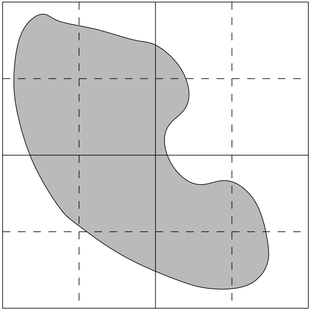
$$s_k \approx s_{k-1} + 2 \cdot r_{k-1} \quad \Rightarrow \quad s_k \approx \sum_{\kappa=1}^{k-1} 2 r_\kappa = 2 \cdot (2^{k+1} - 1).$$

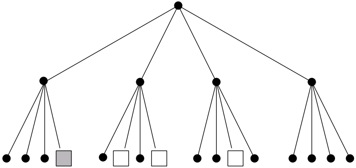
Number of Quadtree Cells to Store a Rectangle



- uniformly refined voxel-grid of level k: $\left(2^k\right)^2 =: \mathcal{O}(N^2)$ cells
- octree-refined grid of level k: $r_k + s_k \approx 3 \cdot 2^{k+1} = 6 \cdot 2^k$ cells
  $\Rightarrow$ number of cells $\mathcal{O}(N)$, i.e. proportional to length of boundary

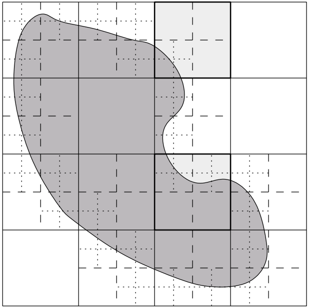
Storing a Quadtree – Sequentialisation

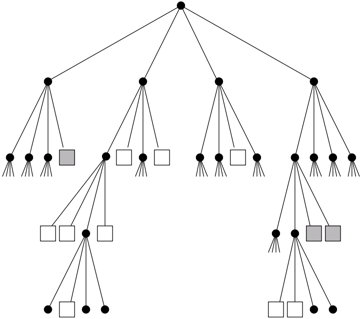




- sequentialise cell information according to *depth-first traversal*
- local numbering of the child nodes on each level determines sequential order

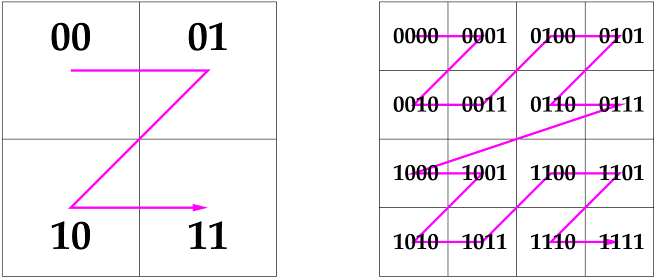
Storing a Quadtree – Sequentialisation





- sequentialise cell information according to *depth-first traversal*
- local numbering of the child nodes on each level determines sequential order
- here: leads to so-called **Morton order** (for uniform grids)

Morton Order



Relation to bit arithmetics:

- odd digits: position in vertical direction
- even digits: position in horizontal direction

Morton Order and Cantor's Mapping

Georg Cantor (1877):

$$0.01111001\ldots \rightarrow \begin{pmatrix} 0.0110\ldots \\ 0.1101\ldots \end{pmatrix}$$

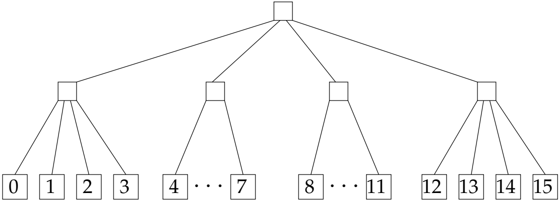
- **bijective** mapping $[0,1] \rightarrow [0,1]^2$
- proved identical cardinality of $[0,1]$ and $[0,1]^2$
- provoked the question: is there a **continuous** mapping? (i.e. a curve)

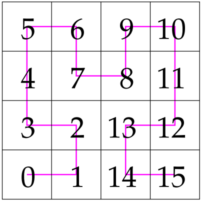
Preserving Neighbourship for a 2D Octree

Requirements:

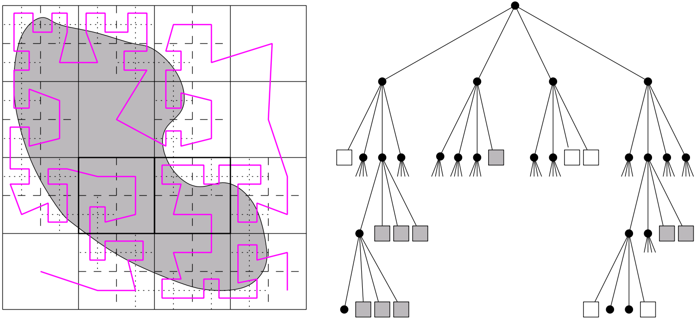
- consider a simple 4×4-grid
- uniformly refined
- subsequently numbered cells should be neighbours in 2D

Leads to (more or less unique) numbering of children:



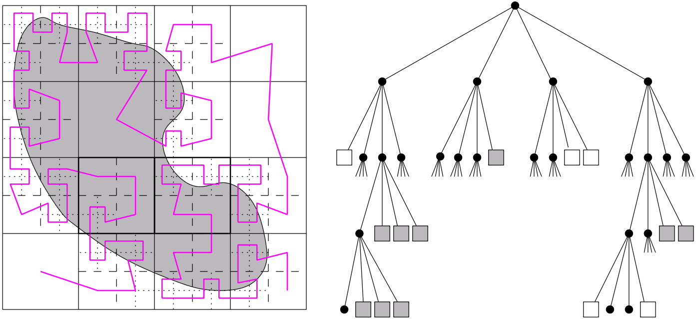


Preserving Neighbourship for a 2D Octree (2)



- adaptive refinement possible
- neighbours in sequential order remain neighbours in 2D

Preserving Neighbourship for a 2D Octree (2)



- adaptive refinement possible
- neighbours in sequential order remain neighbours in 2D
- here: similar to the concept of **Hilbert curves**

Open Questions

Algorithmics:

- How do we describe the sequential order algorithmically?
- What kind of operations are possible?
- Are there further “orderings” with the same or similar properties?

Applications:

- Can we quantify the “neighbour” property?
- In what applications can this property be useful?
- What further operations are required or possible?